Some Fuss About Fusion
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§1. Some topological motivation

Let G, G' be finite groups with Sylow p-subgroups S, S'.

Theorem: (Cartan-Eilenberg) $\mathrm{H}^*(G;\mathbb{F}_p)$ is isomorphic to the subring of stable elements in $\mathrm{H}^*(S,\mathbb{F}_p)$, where we recall that an element $a \in \mathrm{H}^*(G;\mathbb{F}_p)$ is *stable* if

$$\begin{array}{ccc} \mathrm{H}^*(S;\mathbb{F}_p) & & \\ \downarrow{\scriptstyle c_g} & \searrow^{\mathrm{res}} & \\ & & \mathrm{H}^*(S\cap S^g;\mathbb{F}_p) \\ & \nearrow_{\mathrm{res}} & \\ \mathrm{H}^*(S^g;\mathbb{F}_p) & & \end{array}$$

$\mathrm{res}(a) = (\mathrm{res}\circ c_g)(a)$ for all $g \in G$.

This says that the mod p cohomology of a finite group is controlled by p-fusion.

Theorem: (Martino-Priddy, Oliver) The following are equivalent:

1. $BG^\wedge_p \simeq BG'^\wedge_p$
2. There exists a (G,G')-fusion preserving isomorphism $S \xrightarrow{\cong} S'$, where an isomorphism $\phi : S \xrightarrow{\cong} S'$ is (G,G')-fusion preserving if

$$\begin{array}{ccc} G & & G' \\ \uparrow & & \uparrow \\ S & \xrightarrow[\phi]{\cong} & S' \end{array}$$

$$P \xrightarrow[\alpha]{\cong} Q \qquad \phi(P) \xrightarrow[\phi\circ\alpha\circ\phi^{-1}]{\cong} \phi(Q)$$

whenever $\alpha : P \to Q$ is an isomorphism of subgroups of S, then

$$\alpha = c_g|_P \text{ for some } g \in G \iff \phi\circ\alpha\circ\phi^{-1} = c_{g'}|_{\phi(P)} \text{ for some } g' \in G'.$$

3. There is an *isotypical* equivalence $\mathcal{F}_p(G) \simeq \mathcal{F}_p(G')$.

This says that the p-local homotopy type of the classifying space a finite group is controlled by p-fusion.

Attached to any finite group is the fusion system $\mathcal{F}_p(G)$ and the centric linking system $\mathcal{L}_p^c(G)$, which are related by

$$|\mathcal{L}_p^c(G)|_p^\wedge \simeq BG_p^\wedge.$$

The goal is then to build an algebraic object for which "classifying space" makes sense and for which one can compute the p-local homotopy type of the classifying space from the algebraic structure of the object. So generally, we start with an abstract fusion system $\mathcal{F}$ on a finite p-group S and a centric linking system $\mathcal{L}$ corresponding to $\mathcal{F}$. This data makes up a p-local finite group $(S, \mathcal{F}, \mathcal{L})$. We define the classifying space of $(S, \mathcal{F}, \mathcal{L})$ to be $|\mathcal{L}|_p^\wedge$.

Using the Martino-Priddy conjecture, one can show that if S is a Sylow p-subgroup of G and $(S, \mathcal{F}_S(G), \mathcal{L})$ is a p-local finite group, then $\mathcal{L}$ is equivalent to $\mathcal{L}_S^c(G)$. In general it is not known whether given S and $\mathcal{F}$, one can find $\mathcal{L}$ such that $(S, \mathcal{F}, \mathcal{L})$ is a p-local finite group. If such an $\mathcal{L}$ exists, it is not known whether it is unique. However, it is known that given S and $\mathcal{F}$, there is a bijection between the centric linking systems associated to $\mathcal{F}$ and the classifying spaces for $\mathcal{F}$ (up to appropriate equivalences).

We can define the cohomology of a fusion system, $\mathrm{H}^*(\mathcal{F}; \mathbb{F}_p)$ as the subring of $\mathrm{H}^*(BS; \mathbb{F}_p)$ consisting of $\mathcal{F}$-stable elements. Then one obtains

$$\mathrm{H}^*(|\mathcal{L}|_p^\wedge; \mathbb{F}_p) \cong \mathrm{H}^*(\mathcal{F}; \mathbb{F}_p).$$

Interesting??? Well, there might be a fusion system associated to each block of a finite group. This means that there might be an associated classifying space for any block. Actually, the fusion system of many different blocks is known to come from a finite group, but not all.

§2. Fusion Systems of Blocks

Let G be a finite group with normal subgroup N, p a prime, k an algebraically closed field of characteristic p, c a G-stable p-block of kN and b a p-block of kG covering c, i.e., $bc = b$.

$$\begin{array}{ccc} G & & b \\ \Vert & & \downarrow \\ N & & c \end{array}$$

Note that it may be the case that c has fusion and b does not, i.e., b is nilpotent, and c is not.

Definition (Broué-Puig):

1. Let Q be a p-subgroup of G. The **Brauer homomorphism** (with respect to N and Q), Br_Q^N is the restriction of the projection map
$$kN \to k\mathrm{C}_N(Q)$$
to $(kN)^Q$, the Q-fixed points of kN.

2. Let $\mathscr{S}_p(G)$ denote the set of p-subgroups of G and set
$$\mathscr{S}_p(G,c) := \{(Q,e) \mid Q \in \mathscr{S}_p(G) \text{ and } e \in \mathrm{Bl}(k\mathrm{C}_N(Q)) \text{ such that } \mathrm{Br}_Q^N(c)e \neq 0.\}$$
These are called (c,G)-**Brauer pairs**

3. There is a notion of inclusion of Brauer pairs and a form of Sylow's theorem holds. We denote the set of maximal elements in $\mathscr{S}_p(G,c)$ by $\mathrm{Syl}_p(G,c)$.

Remark: If $N = G$, then $\mathscr{S}_p(G,c) = \mathscr{S}_p(G,b)$ are just the usual b-Brauer pairs and if $(P,e) \in \mathrm{Syl}_p(G,b)$ then P is a defect group of b. If b is the principal block, then there is a bijection $\mathscr{S}_p(G,b) \simeq \mathscr{S}_p(G)$.

Definition and Theorem (Kessar-Stancu):

1. Let $(S,e_S) \in \mathrm{Syl}_p(G,c)$ and for $Q \leq S$, let e_Q be the unique block of $k\mathrm{C}_N(Q)$ such that $(Q,e_Q) \leq (S,e_S)$. Define a category on S:
$$\mathcal{F}_{(S,e_S)}(G,N,c) := \begin{cases} \text{objects: } Q \leq S \\ \mathrm{Hom}_{\mathcal{F}_{(S,e_S)}(G,N,c)}(Q,R) = \{c_g : Q \to R \mid g \in G, (Q,e_Q)^g \leq (R,e_R)\} \end{cases}$$
Then $\mathcal{F}_{(S,e_S)}(G,N,c)$ is a fusion system on S. In case $N = G$, we denote $\mathcal{F}_{(S,e_S)}(G,N,c)$ by $\mathcal{F}_{(S,e_S)}(G,b)$.

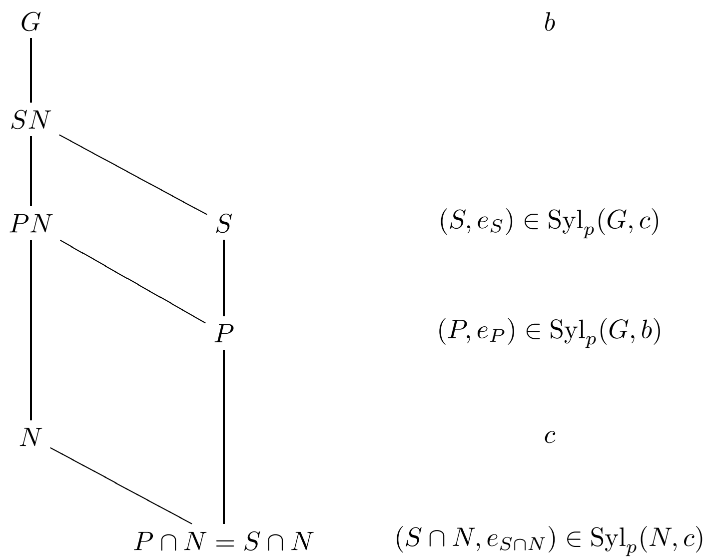
2. Let $(P,e_P) \in \mathrm{Syl}_p(G,b)$.

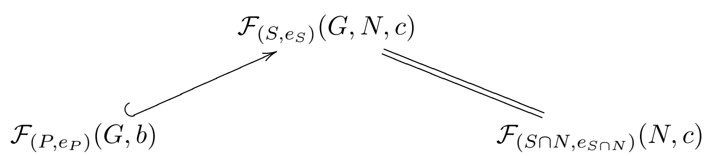
(a) There exists $(S, e_S) \in \mathrm{Syl}_p(G, c)$ such that $(P, e_P) \leq (S, e_S)$ and such that $\mathcal{F}_{(P,e_P)}(G, b) \subseteq \mathcal{F}_{(S,e_S)}(G, N, c)$.

(b) $P \cap N = S \cap N$.

(c) $(S \cap N, e_{S\cap N}) \in \mathrm{Syl}_p(N, c)$

(d) $\mathcal{F}_{(S\cap N, e_{S\cap N})}(N, c) \lhd \mathcal{F}_{(S,e_S)}(G, N, c)$.





§3. Control of Fusion

In this section, we generalize some results of Glauberman and Thompson to arbitrary fusion systems. As a consequence, we get new results about blocks.

Definition: Let S be a finite p-group. The **Thompson subgroup** of S, denoted $J(S)$, is the subgroup generated by the abelian subgroups of S of maximal order.

Let $\mathcal{F}$ be a fusion system on a finite p-group S.

Theorem (Kessar-Linckelmann, K-L-Robinson): Let p be odd. Then

$$\mathcal{F} = \mathcal{F}_S(S) \iff \mathrm{N}_{\mathcal{F}}(Z(J(S))) = \mathcal{F}_S(S).$$

Here $\mathrm{N}_{\mathcal{F}}(\mathrm{Z}(\mathrm{J}(S)))$ is the subfusion system of $\mathcal{F}$ on S containing all morphisms in $\mathcal{F}$ that *stably extend* to $\mathrm{Z}(\mathrm{J}(S))$, i.e., if $\phi \in \mathrm{Hom}_{\mathrm{N}_{\mathcal{F}}(\mathrm{Z}(\mathrm{J}(S)))}(Q, R)$, then there exists $\widetilde{\phi} \in \mathrm{Hom}_F(Q\mathrm{Z}(\mathrm{J}(S)), R\mathrm{Z}(\mathrm{J}(S)))$ such that $\widetilde{\phi}|_Q = \phi$ and $\widetilde{\phi}(\mathrm{Z}(\mathrm{J}(S))) = \mathrm{Z}(\mathrm{J}(S))$.

Theorem (Díaz, Glesser, Mazza, Park): Let p be odd. Then

1. $\mathrm{Z}(\mathcal{F}) = \mathrm{Z}(\mathrm{N}_{\mathcal{F}}(\mathrm{J}(S)))$.
   Here $\mathrm{Z}(\mathcal{F}) = \{x \in S \mid \text{if } \phi \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, S) \text{ then } \phi(x) = x\}$.

2. $\mathcal{F} = \mathcal{F}_S(S) \iff \mathrm{C}_{\mathcal{F}}(\mathrm{Z}(S)) = \mathcal{F}_S(S) = \mathrm{N}_{\mathcal{F}}(\mathrm{J}(S))$.
   Here $\mathrm{C}_F(\mathrm{Z}(S))$ is the subfusion system of $\mathcal{F}$ on S containing all morphisms in $\mathcal{F}$ that *centrally extend* to $\mathrm{Z}(S)$, i.e., if $\phi \in \mathrm{Hom}_{\mathrm{C}_{\mathcal{F}}(\mathrm{Z}(S))}(Q, R)$, then there exists $\widetilde{\phi} \in \mathrm{Hom}_F(Q\mathrm{Z}(\mathrm{J}(S)), R\mathrm{Z}(\mathrm{J}(S)))$ such that $\widetilde{\phi}|_Q = \phi$ and $\widetilde{\phi}|_{\mathrm{Z}(\mathrm{J}(S))} = \mathrm{id}_{\mathrm{Z}(\mathrm{J}(S))}$.

3. The following are equivalent:
   (a) $\mathcal{F} = \mathcal{F}_S(S)$.
   (b) $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a p-group for every nontrivial characteristic subgroup Q of S.
   (c) $\mathrm{N}_{\mathcal{F}}(Q) = \mathcal{F}_S(S)$ for every nontrivial characteristic subgroup Q of S.

   This is a generalization of Frobenius' p-nilpotentcy criterion. Also, there is a notion of a fusion system being H-free for a finite group H akin to the notion of a group being H-free. In the above theorem, the results hold when p is odd or when $\mathcal{F}$ is S_4-free.

Defintion: The $\mathcal{F}$**-focal subgroup** of S is

$$[S,\mathcal{F}] := \langle u^{-1}\phi(u) \mid u \in S, \phi \in \mathrm{Hom}_{\mathcal{F}}(\langle u \rangle, S)\rangle.$$

A subfusion system $\mathcal{G}$ of $\mathcal{F}$ on S **controls transfer** in $\mathcal{F}$ if

$$[S,\mathcal{G}] = [S,\mathcal{F}]$$

Glauberman defined characteristic subgroups K^∞ and K_∞ of any finite p-group

Theorem (DGMP): Assume $p > 3$. Then $\mathrm{N}_{\mathcal{F}}(K^\infty(S))$ and $\mathrm{N}_{\mathcal{F}}(K_\infty(S))$ control transfer in $\mathcal{F}$.